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Nearest Points and Some Fixed Point Theorems for Weakly Compact Sets[1]

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We prove the following theorem and then derive several known results as corollaries. Let X be a convex subset of E and let $f: (X, w) \to (E, T)$ be a continuous mapping. Let S be a nonempty, convex, and w-compact subset of X and K a w-compact subset of X. Let $p \in \mathbf{P}$ and let f satisfy the condition: for each $y \in X \setminus K$, there exists a $x \in S$ such that $p(x - fy) < p(y - fy)$. Then there exists a $u \in X$ satisfying $p(u - fu) = \min\{p(x - fu) : x \in X\}$. © 1987 Academic Press, Inc.

Let (E, T) be a locally convex topological vector space with topology T and $E^* = (E, T)^*$ be its topological dual. Let $w = w(E, E^*)$ be the weak topology of E and let P and Q denote the family of continuous seminorms generating the topologies T and w, respectively. Clearly $Q \subseteq P$ since $w \subseteq T$. We prove the following theorem and then derive several results as corollaries.

THEOREM 1. *Let X be a convex subset of E and let $f: (X, w) \to (E, T)$ be a continuous mapping. Let S be a nonempty, convex, and w-compact subset of X and K a w-compact subset of X. Let $p \in P$ and let f satisfy the condition: for each $y \in X \setminus K$, there exists a $x \in S$ such that*

$$p(x - fy) < p(y - fy). \tag{1}$$

Then there exists a $u \in X$ satisfying

$$p(u - fu) = \min\{p(x - fu) : x \in X\}. \tag{2}$$

We first give some consequences of Theorem 1. Corollary 1 is a slight

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extension of the result in [6] and it also improves a well-known result of Ky Fan [2] (see Corollary 2 below).

COROLLARY 1. *Let X be a nonempty convex and w-compact subset of E and let $f\colon (X, w) \to (E, T)$ be a continuous mapping. Then for each $p \in P$, there exists a $u \in X$ satisfying* (2).

Proof. Let $X = K = S$ in Theorem 1. Since in this case $X \setminus K = \emptyset$, condition (1) is trivially satisfied. Consequently the result follows by Theorem 1.

Remark. It may be remarked that if E is additionally a Hausdorff space in Corollary 1, then it also follows that either f has a fixed point or there exist a $p \in P$ and a $u \in X$ with $0 < p(u - fu) = \min\{p(x - fu)\colon x \in X\}$.
The proof is similar to Corollary 1 in [6].

COROLLARY 2 (Ky Fan). *Let X be a nonempty convex and compact subset of (E, T) and let $f\colon (X, T) \to (E, T)$ be a continuous mapping. Then for each $p \in P$, there exists a $u = u(p)$ satisfying* (2).

Proof. It suffices to show that $f\colon (X, w) \to (E, T)$ is continuous. Let C be a T-closed subset of E. Then by hypothesis $f^{-1}(C)$ is T-closed in X and hence is T-compact. This implies that $f^{-1}(C)$ is w-compact and since w-topology is Hausdorff, it follows that $f^{-1}(C)$ is w-closed.

Now we give the following lemma to be used in the proof of our theorem.

LEMMA 1. *Let X be a subset of E and let $f\colon (X, w) \to (E, T)$ be a continuous mapping. Then for any $p \in P$ and a fixed $x \in E$, the mapping $g\colon X \to R$ (reals) defined by*

$$g(y) = p(y - fy) - p(x - fy) \tag{3}$$

is w-lsc (lower semicontinuous).

Proof. Note that the function g is lsc on (X, w) (see [4]) iff for each real r, the set $A = \{y \in X\colon g(y) \leqslant r\}$ is w-closed. Let $\{y_\alpha\colon \alpha \in \Delta\}$ be a net in A and a $y \in E$ such that $y_\alpha \to y$ in w-topology. Then by hypothesis $p(x - fy_\alpha) \to p(x - fy)$. Choose a $x^* \in E^*$ by the Hahn–Banach theorem (see [5]) such that $x^*(y - fy) = p(y - fy)$ while $x^*(z) \leqslant p(z)$ for each $z \in E$. Since $y_\alpha - fy_\alpha \to y - fy$ in w-topology, it follows that

$$p(y - fy) - p(x - fy) = \lim_\alpha (x^*(y_\alpha - fy_\alpha) - p(x - fy_\alpha)). \tag{4}$$

However, for each $\alpha \in \Delta$,

$$x^*(y_\alpha - fy_\alpha) - p(x - fy_\alpha) \leqslant p(y_\alpha - fy_\alpha) - p(x - fy_\alpha) \leqslant r.$$

Consequently, it follows by (4) that

$$g(y) \leqslant r,$$

that is, the set A is w-closed.

The following result due to Allen [1] will be used in the proof of Theorem 1.

THEOREM 2 (Allen [1]). *Let X be a nonempty convex set of a topological vector spacce E and let g be a real valued function defined on $X \times X$ such that*

(i) *for each fixed $x \in X$, $g(x, y)$ is lsc function of y on X,*

(ii) *for each fixed $y \in X$, $g(x, y)$ is quasi-concave function of x on X,*

(iii) *$g(x, x) \leqslant 0$ for all $x \in X$.*

(iv) *X has a nonempty compact, convex subset S such that $\{y \in X: g(x, y) \leqslant 0$ for all $x \in S\}$ is compact.*

Then there exists a point $u \in X$ with $g(x, u) \leqslant 0$ for all $x \in X$.

Recall that a real valued function f defined on a convex set X is quasi-concave iff for each real t, the set

$$\{x \in X: f(x) > t\} \qquad \text{is convex.}$$

Proof of Theorem 1. Define a mapping G: $X \times X \to R$ by

$$G(x, y) = p(y - fy) - p(x - fy).$$

We show that G satisfies the conditions of Theorem 2. Condition (1) follows by Lemma 1. Further, since for any y, and x_1, $x_2 \in X$, $a \geqslant 0$, $b \geqslant 0$ with $a + b = 1$,

$$p(ax_1 + bx_2 - fy) \leqslant ap(x_1 - fy) + bp(x_2 - fy),$$

it follows that for a fixed $y \in X$, $G(x, y)$ is a quasi-concave function of x on X. Condition (iii) holds trivially. To prove (iv) let

$$A = \{y \in X: G(x, y) \leqslant 0 \quad \text{for all} \quad x \in S\}.$$

If $A = \phi$, then A is w-compact subset of S. If $A \neq \phi$ and $y \in A$, then

$$p(y - fy) \leqslant p(x - fy) \qquad \text{for all} \quad x \in S.$$

In view of (1) this implies that $y \in K$. Thus $A \subseteq K$. Further, A is w-closed. In fact, for any fixed $x \in S$, by Lemma 1, we have

$$G(x, y) = g(y) = p(y - fy) - p(x - fy)$$

is w-lsc.

This implies that the set $\{y: G(x, y) \leqslant 0\}$ is a w-closed subset of K. Consequently,

$$A = \bigcap_{x \in S} \{y: G(x, y) \leqslant 0\}$$

is w-closed subset of K. Since K is w-compact, it follows that A is w-compact. Thus, conditions of Theorem 2 are satisfied for (E, w). Hence for some $u \in X$,

$$p(u - fu) \leqslant p(x - fu)$$

for all $x \in X$.

In the setting of a normed linear space, we have the following.

THEOREM 3. *Let X be a convex subset of a normed linear space E and let $f: (X, w) \to (E, \| \ \|)$ be a continuous mapping. Let S be a nonempty weakly compact, convex subset of X, and K a weakly compact subset of X. If f satisfies the condition: for each $y \in X \backslash K$, there exists a $x \in S$ such that*

$$\|x - fy\| < \|y - fy\|$$

then there exists a $u \in X$ with $\|u - fu\| = \min\{\|x - fu\|: x \in X\}$.

It may be pointed out that Theorem 3, and consequently Theorem 1, provides an extension of a recent result of Ky Fan (see [3, Theorem 7]).

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